Sets avoiding integral distances

SASCHA KURZ
Department of Mathematics
University of Bayreuth
D-95440 Bayreuth, Germany
sascha.kurz@uni-bayreuth.de

VALERY MISHKIN
Department of Mathematics and Statistics
York University
Toronto ON, M3J1P3 Canada
vmichkin@mathstat.yorku.ca

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Abstract

We study open point sets in Euclidean spaces $\mathbb{R}^d$ without a pair of points an integral distance apart. By a result of Furstenberg, Katznelson, and Weiss such sets must be of Lebesgue upper density zero. We are interested in how large such sets can be in d-dimensional volume. We determine the lower and upper bounds for the volumes of the sets in terms of the number of their connected components and dimension, and also give some exact values. Our problem can be viewed as a kind of inverse to known problems on sets with pairwise rational or integral distances.

1 Introduction

Is there a dense set S in the plane so that all pairwise Euclidean distances between the points are rational? This famous open problem was posed by Ulam in 1945, see e.g. [17, 18, 38].[1] If all pairwise distances between points in S are integral and S is non-collinear, i.e. not all points are located on a line, then S is finite [2, 16]. Having heard of this result, Ulam guessed that the answer to his question would be in the negative. Of course the rational numbers form a dense subset of a coordinate line with pairwise rational distances; also, on a circle there are dense sets with pairwise rational distances, see e.g. [1, 2]. It has been proved by Solymosi and De Zeeuw [36] that the line and the circle are the only two irreducible algebraic curves containing infinite subsets of points with pairwise rational distances.[2] There is interest in a general construction of a planar point set $S(n,k)$ of size n with pairwise integral distances such that $S(n,k) = A \cup B$ where A is collinear, $|A| = n-k$, $|B| = k$, and B has no three collinear points. The current record is $k = 4$ [10]. And indeed, it is very hard to construct a planar point set, no three points on a line, no four points on a circle, with pairwise integral distances. Kreisel and Kurz [28] found such a set of size 7, but it is unknown if there exists one of size 8.

The present paper is concerned with a problem that may be considered as an inverse to those just described, namely with *large* point sets in $\mathbb{R}^d$ without a pair of points an integral distance apart. We write $f_d(n)$ for the supremum of the volumes $\lambda_d(\mathcal{P})$ of open point sets $\mathcal{P} \subset \mathbb{R}^d$ with n connected components without a pair of points whose distance apart is a positive integer. We give general bounds for $f_d(n)$ and determine the exact values in several cases.

[1] A construction of a countable dense set in the plane avoiding rational distances is provided in the Appendix.
[2] Point sets with rational coordinates on spheres have been considered in [33].

This problem is related to the famous Hadviger-Nelson open problem of determining the (measurable) chromatic number of $\mathbb{R}^d$, see e.g. [12, Problem G10]. Here one can also ask for the highest density of one color class in such a coloring, that is, we may ask for the densest set without a pair of points a distance 1 apart. In [30] such a construction in $\mathbb{R}^3$ has been given. In the plane the best known example, due to Croft [11], consists of the intersections of hexagons with circles and attains a density of 0.2294. The upper bounds are computed in [5, 13]. Point sets avoiding a finite number k of prescribed distances are considered e.g. in [9] and [12, Problem G4], so the point sets avoiding all distances that are positive integers correspond to the case with an infinite number κ of excluded distances. It is known [21] that for each subset of the plane with positive upper density δ, there is a constant $d(\delta)$ such that the distances larger than d are unavoidable. The same result is true in higher dimensions [32]. It follows that in every dimension $d \geq 2$, the Lebesgue measurable sets avoiding integral distances, which are of interest here, must be of upper density zero, so we consider the supremum of their volumes instead.

The paper is organized as follows: in Section 2 we find the upper bounds for the volumes of some sets with the above mentioned property in dimension 1 and higher. To this end we consider sets with a slightly weaker property. We denote by $l_d(n)$ the least upper bound of the volumes of d-dimensional open set with n connected components, each of diameter at most one, whose intersection with every line have total lengths of at most one. In this context we mention the closely related field of geometric tomography, see e.g. [22]. In order to obtain tighter bounds we consider special cases where the connected components are all d-dimensional open balls. In Section 3 several constructions are provided for the four classes of problems introduced. In Section 4 we give a summary of the results obtained and draw the appropriate conclusions. Some additional computations are provided in the Appendix.

2 Upper bounds

Denote by $\text{dist}(x, y)$ the Euclidean distance between two points $x, y \in \mathbb{R}^d$ and by $\text{dist}(V, W) := \inf\{\text{dist}(x, y) \mid x \in V, y \in W\}$ the distance between two subsets V and W of $\mathbb{R}^d$. The minimal width of V, i.e. the minimum distance between parallel support hyperplanes of the closed convex hull of V, will be denoted by $\text{width}(V)$, and λ_d will stand for the Lebesgue measure in $\mathbb{R}^d$.

At first we observe that the diameter of any connected component of an open set avoiding integral distances, i.e. having no points an integral distance apart, is at most one.

Lemma 1 *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be an open set avoiding integral distances. Then for every connected component $\mathcal{C}$ of $\mathcal{P}$ we have diam$(\mathcal{C}) \leq 1$.*

PROOF. Suppose there is a connected component $\mathcal{C}$ with $\text{diam}(\mathcal{C}) > 1$, then there exist $x_1, x_2 \in \mathcal{C}$ such that $\text{dist}(x_1, x_2) > 1$. Since $\mathcal{C}$ is open, it is path connected, so there is a point x on the image curve of a continuous path in $\mathcal{C}$ joining x_1 and x_2 such that $\text{dist}(x_1, x) = 1$. □

By the isodiametric inequality the open ball $B_d \subset \mathbb{R}^d$ centered at the origin with unit diameter has the largest volume among measurable sets in $\mathbb{R}^d$ of diameter at most 1, see e.g. [19], [6, chap. 2]. Thus we have

$$f_d(1) = \lambda_d(B_d) = \frac{\pi^{d/2}}{2^d \cdot \Gamma\left(\frac{d}{2}+1\right)} = \begin{cases} \frac{\pi^{\frac{d}{2}}}{2^d \left(\frac{d}{2}\right)!} & \text{for } d \text{ even,} \\ \frac{\left(\frac{d-1}{2}\right)! \cdot \pi^{\frac{d-1}{2}}}{d!} & \text{for } d \text{ odd.} \end{cases}$$

The first few values are given by $\lambda_1(B_1) = 1$, $\lambda_2(B_2) = \frac{\pi}{4}$, $\lambda_3(B_3) = \frac{\pi}{6}$, and $\lambda_4(B_4) = \frac{\pi^2}{32}$.

Next we characterize 1-dimensional open sets containing a pair of points an integral distance apart.

Lemma 2 *A non-empty open set $\mathcal{P} \subseteq \mathbb{R}$ contains a pair of points $x, y \in \mathcal{P}$ with dist$(x, y) \in \mathbb{N}$ if and only if either $\lambda_1(\mathcal{P}) > 1$ or there is a pair of connected components (i.e. disjoint open intervals) $\mathcal{C}_1$, $\mathcal{C}_2$ of $\mathcal{P}$ such that dist$(\mathcal{C}_1, \mathcal{C}_2) \notin \mathbb{N}$ and $\lambda_1(\mathcal{C}_1 \cup \mathcal{C}_2) > \lceil \text{dist}(\mathcal{C}_1, \mathcal{C}_2) \rceil - \text{dist}(\mathcal{C}_1, \mathcal{C}_2)$. If $\lambda_1(\mathcal{P}) \leq 1$, then there exists a shift $f : x \mapsto x + a$ of $\mathbb{R}$ such that $f(\mathcal{P}) \cap \mathbb{Z} = \varnothing$.*

PROOF. The restriction of the canonical epimorphism $\phi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, $x \mapsto x + \mathbb{Z} = (x - \lfloor x \rfloor) + \mathbb{Z}$, to the interval $[0, 1)$ is a continuous bijection of $[0, 1)$ onto the 1-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the inverse map $\phi|_{[0,1)}^{-1}$ being continuous at all points except $\phi(0) = 0 + \mathbb{Z} = \mathbb{Z} \in \mathbb{T}$. We consider the retraction $\phi_1 := \phi|_{[0,1)}^{-1} \circ \phi : \mathbb{R} \to [0, 1)$, that is, $\phi_1(x) = x - \lfloor x \rfloor$ for all $x \in \mathbb{R}$ (i.e. $\phi_1(x) = x \bmod 1$ is the fractional part of x). We observe that the image under ϕ_1 of any open interval (x, y) of length $y - x < 1$ is either the open interval $(\phi_1(x), \phi_1(y)) = (x - n, y - n)$ of the same length $\phi_1(y) - \phi_1(x) = (x - n) - (y - n) = y - x$, whenever both x and y are in $(n, n + 1)$, for some $n \in \mathbb{Z}$, or the union of two disjoint connected components

$$[0, \phi_1(y)) \cup (\phi_1(x), 1) = [0, y - n) \cup (1 - (n - x), 1)$$

of the same total length $(y - n) + (n - x) = y - x$, whenever $x < n < y$, for some $n \in \mathbb{Z}$. If $y - x = 1$, then similarly either $\phi_1((n, n+1)) = (0, 1)$ or $\phi_1((x, y)) = [0, y-n) \cup (1-(n-x), 1) = [0, 1) \setminus \{y-n\}$ whenever $x < n < y$ for some $n \in \mathbb{N}$. Hence, in general, the total length of the connected components of $\phi_1((x, y))$ is $y - x$, whenever $y - x \leq 1$.

Let $\mathcal{P}$ be the disjoint union of open intervals $\mathcal{C}_i$, say, with total length $\lambda_1(\mathcal{P}) = \sum_i \lambda_1(\mathcal{C}_i) > 1$. Then by Lemma 1 $i \geq 2$ and $\lambda_1(\mathcal{C}_i) \leq 1$ for all i. We thus have from above that the total length of the connected components of all the images $\phi_1(\mathcal{C}_i)$ equals $\sum_i \lambda_1(\mathcal{C}_i) > 1$. Hence at least two images $\phi_1(\mathcal{C}_k)$ and $\phi_1(\mathcal{C}_j)$ must overlap, so there exists $z \in \phi_1(\mathcal{C}_k) \cap \phi_1(\mathcal{C}_j)$, that is, $x_0 - \lfloor x_0 \rfloor = y_0 - \lfloor y_0 \rfloor$ for some $x_0 \in \mathcal{C}_k$ and $y_0 \in \mathcal{C}_j$. Thus $x_0 - y_0 = \lfloor x_0 \rfloor - \lfloor y_0 \rfloor \in \mathbb{Z} \setminus \{0\}$, hence $\mathrm{dist}(x_0, y_0) \in \mathbb{N}$.

If $\lambda_1(\mathcal{C}_1 \cup \mathcal{C}_2) > \alpha$ for some connected components $\mathcal{C}_1 = (a, b)$ and $\mathcal{C}_2 = (c, d)$ of $\mathcal{P}$ with $\mathrm{dist}(\mathcal{C}_1, \mathcal{C}_2) = c - b = m - \alpha$, where $m \in \mathbb{N}$, $0 < \alpha < 1$, so that $\lceil \mathrm{dist}(\mathcal{C}_1, \mathcal{C}_2) \rceil - \mathrm{dist}(\mathcal{C}_1, \mathcal{C}_2) = \alpha$, we can take a point x in the leftmost interval, say $x \in \mathcal{C}_1$ and a point $y \in \mathcal{C}_2$ so that the length of $(x, b) \cup (c, y)$ is $\alpha < \lambda_1(\mathcal{C}_1 \cup \mathcal{C}_2) = (b - a) + (d - c)$. Then

$$\mathrm{dist}(x, y) = (b - x) + m - \alpha + (y - c) = \alpha + m - \alpha = m \in \mathbb{N}.$$

Conversely, suppose there are $x, y \in \mathcal{P}$ with $\mathrm{dist}(x, y) = k \in \mathbb{N}$. If x and y lie in the same connected component $\mathcal{C}_i$ of $\mathcal{P}$, then $\lambda_1(\mathcal{C}_i) > k \geq 1$ because $\mathcal{C}_i$ is open, hence $\lambda_1(\mathcal{P}) > 1$. Let x and y lie in distinct connected components of $\mathcal{P}$, say $x < y$ and $x \in \mathcal{C}_1 = (a, b)$, $y \in \mathcal{C}_2 = (c, d)$, and let $\lambda_1(\mathcal{P}) \leq 1$. Then $(b - a) + (d - c) \leq 1$ as well whence the distance between the components is $\mathrm{dist}(\mathcal{C}_1, \mathcal{C}_2) = c - b \notin \mathbb{N}$, because $c - b < \mathrm{dist}(x, y) < c - b + [(b - a) + (d - c)] < c - b + 1$. Let $c - b = m - \alpha$ where $m \in \mathbb{N}$, $0 < \alpha < 1$. Then

$$\alpha = m + b - c < m + 1 + b - c < (d - a) + (b - c) = (b - a) + (d - c) = \lambda_1(\mathcal{C}_1 \cup \mathcal{C}_2),$$

since $m + 1 < d - a$ because $m + 1 \leq k < d - a$. Thus either $\lambda_1(\mathcal{P}) > 1$ or there is a pair of required connected components of $\mathcal{P}$.

If $\lambda_1(\mathcal{P}) \leq 1$, then $\lambda_1(\mathcal{C}_i) \leq 1$ for all i, so the total length of the connected components of all the images $\phi_1(\mathcal{C}_i)$ equals $\sum_i \lambda_1(\mathcal{C}_i) = \lambda_1(\mathcal{P})$, as we have shown above. If $\lambda_1(\mathcal{P}) < 1$, then clearly, $\phi_1(\mathcal{P}) \neq [0, 1)$. If $\lambda_1(\mathcal{P}) = 1$, then again $\phi_1(\mathcal{P}) \neq [0, 1)$, whenever the images $\phi_1(\mathcal{C}_i)$ are not pairwise disjoint. Suppose all the images $\phi_1(\mathcal{C}_i)$ are pairwise disjoint and $\mathcal{P} \cap \mathbb{Z} \neq \varnothing$. Then there is exactly one $\mathcal{C}_j = (a, b)$ that meets $\mathbb{Z}$. Hence the complement $[0, 1) \setminus \phi_1(\mathcal{C}_j) = [\phi_1(b), \phi_1(a)]$ is a non-open set in $\mathbb{R}$ that can not be covered by the images $\phi_1(\mathcal{C}_i)$ of the other connected components of $\mathcal{P}$, since they are all open intervals, so $\phi_1(\mathcal{P}) \neq [0, 1)$ as well. Thus in all the cases we have $\phi_1(\mathcal{P}) \neq [0, 1)$. Take $\phi_1(a) \in [0, 1) \setminus \phi_1(\mathcal{P})$, $a \in \mathbb{R}$, that is, $\phi_1(a) \cap \phi_1(\mathcal{P}) = \varnothing$. Then $\phi(a) \cap \phi(\mathcal{P}) = \varnothing$, i.e. $(a + \mathbb{Z}) \cap (\mathcal{P} + \mathbb{Z}) = \varnothing$, so $(\mathcal{P} - a) \cap \mathbb{Z} = \varnothing$ and the required shift is $f : x \mapsto x + (-a)$. □

Applying Lemma 2 we establish a criterion for a point set to avoid integral distances in all dimensions.

Theorem 1 *An open point set $\mathcal{P} \subseteq \mathbb{R}^d$ is free of points with integral distance if and only if for every line $\mathcal{L}$*

(i) $\lambda_1(\mathcal{P} \cap \mathcal{L}) \leq 1$ *and*

(ii) *if $\mathcal{L}$ hits a pair of distinct connected components $\mathcal{C}_1$, $\mathcal{C}_2$ of $\mathcal{P}$ in the intervals $\mathcal{C}_1 \cap \mathcal{L}, \mathcal{C}_2 \cap \mathcal{L}$ with dist$(\mathcal{C}_1 \cap \mathcal{L}, \mathcal{C}_2 \cap \mathcal{L}) = r \notin \mathbb{N}$, then $\lceil r \rceil - r \geq \lambda_1((\mathcal{C}_1 \cup \mathcal{C}_2) \cap \mathcal{L})$.*

As Lemma 1 and Theorem 1.(i) will be our main tools in estimating upper bounds for $f_d(n)$, we denote by $l_d(n)$ the supremum of the volumes $\lambda_d(\mathcal{P})$ of open point sets $\mathcal{P} \subseteq \mathbb{R}^d$ consisting of n connected components with diameters at most 1, where the total length of the intersection with every line is at most 1. Clearly $l_d(1) = f_l(1) = \lambda_d(B_d)$ and $f_d(n) \leq l_d(n)$. We remark that omitting the conditions of Theorem 1 on the line intersections, i.e. assuming only the condition from Lemma 1, leaves over a very simple problem whose extremal configurations consist of n disjoint open d-dimensional balls of diameter 1. Dropping the condition from Lemma 1 on the diameter of the connected components and taking into account only Theorem 1.(i) results in a slightly more challenging problem. It turns out that there are open connected d-dimensional point sets $\mathcal{P}$ with infinite volume $\lambda_d(\mathcal{P})$ and diameter $\mathrm{diam}(\mathcal{P})$ even though the length of the intersection of $\mathcal{P}$ with every line $\mathcal{L}$ is at most 1, i.e. $\lambda_1(\mathcal{P} \cap \mathcal{L}) \leq 1$.

Construction 1 *For integers $n \geq 1$ and $d \geq 2$, denote by $\mathcal{A}_n^d$ the d-dimensional open spherical shell, or annulus, centered at the origin with inner radius n and outer radius $n + \frac{1}{dn^d}$, i.e. $\mathcal{A}_n^d$ are bounded by concentric $(d-1)$-dimensional spheres centered at the origin. Similarly, denote by $\mathcal{B}_n^d$ the d-dimensional open spherical shell centered on the y-axis at $n + \frac{3}{4}$ with inner radius 1 and outer radius $1 + \frac{1}{n^4}$. Then the set $\mathcal{P} = \cup_{n \geq 30} \left(\mathcal{A}_n^d \cup \mathcal{B}_n^d\right)$ is open and connected with infinite volume and diameter even though the length of its intersection with every line is at most 1. In Figure 2 we depicted a configuration for dimension $d = 2$ containing the first few annuli, $\mathcal{A}_n^2$ and $\mathcal{B}_n^2$ being blue and green respectively. The detailed computations demonstrating the assertions claimed are given in the Appendix.*

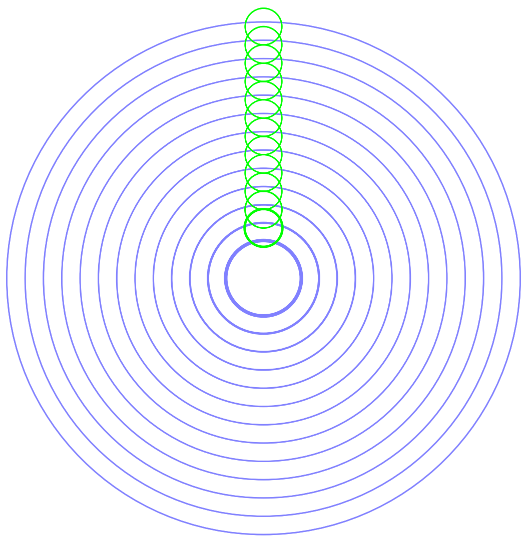

Figure 1: Concentric annuli with infinite area but finite lengths of line intersections.

Proposition 1 *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be distinct connected components of a d-dimensional open point set $\mathcal{P}$ without a pair of points an integral distance apart. If $\lambda_d(\mathcal{C}_1 \cup \mathcal{C}_2) > \lambda_d(B_d)$, then $\mathrm{dist}(\mathcal{C}_1, \mathcal{C}_2) \geq 1$.*

PROOF. By the isodiametric inequality, see the beginning of Section 2, and Lemma 1 we have $\mathrm{diam}(\mathcal{C}_1 \cup \mathcal{C}_2) > 1$, $\mathrm{diam}(\mathcal{C}_1) \leq 1$, and $\mathrm{diam}(\mathcal{C}_2) \leq 1$. Choose $x_1 \in \mathcal{C}_1$, $x_2 \in \mathcal{C}_2$ with $\mathrm{dist}(x_1, x_2) > 1$. If $\mathrm{dist}(\mathcal{C}_1, \mathcal{C}_2) < 1$, then there exist $\bar{x}_1 \in \mathcal{C}_1$ and $\bar{x}_2 \in \mathcal{C}_2$ such that $\mathrm{dist}(\bar{x}_1, \bar{x}_2) < 1$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are open, they are path connected, hence we can join x_1 and $\bar{x}_1$ by a continuous path in $\mathcal{C}_1$ and similarly x_2 and $\bar{x}_2$ in $\mathcal{C}_2$ and on the image curves of these paths we then find $x_1' \in \mathcal{C}_1$ and $x_2' \in \mathcal{C}_2$ such that $\mathrm{dist}(x_1', x_2') = 1$, but $\mathcal{P}$ avoids integral distances, a contradiction. Thus we have $\mathrm{dist}(\mathcal{C}_1, \mathcal{C}_2) \geq 1$. $\square$

Thus in the case where Proposition 1 applies, we can construct a d-dimensional ball with diameter at least 1 touching the closed convex hulls of $\mathcal{C}_1$ and $\mathcal{C}_2$. Indeed, by compactness of the closures $\overline{\mathcal{C}}_1$ and $\overline{\mathcal{C}}_2$ one can find the points $x_1 \in \overline{\mathcal{C}}_1$ and $x_2 \in \overline{\mathcal{C}}_2$ with $\mathrm{dist}(x_1, x_2) = \mathrm{dist}(\mathcal{C}_1, \mathcal{C}_2) \geq 1$. Taking the midpoint of x_1 and x_2 as the center of the ball with diameter $\mathrm{dist}(x_1, x_2)$ yields the required ball disjoint from $\mathcal{P}$.

Lemma 3 *For $d \geq 2$, we have $l_d(2) \leq \lambda_{d-1}(B_{d-1}) \cdot \left(\sqrt{\frac{2d}{d+1}}\right)^{d-1}$.*

PROOF. By Lemma 1 both connected components, denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$, are of diameter at most one, so Jung's theorem [15, 26] yields the enclosing balls $\mathcal{B}_1, \mathcal{B}_2$ for these components of diameter $\sqrt{\frac{2d}{d+1}}$. By Proposition 1 we have $\operatorname{dist}(\mathcal{C}_1, \mathcal{C}_2) \geq 1$, so there is an enclosing cylinder, having a $(d-1)$-dimensional ball of diameter $\sqrt{\frac{2d}{d+1}}$ as its base, containing the closed covex hull $\operatorname{conv}(\overline{\mathcal{B}_1 \cup \mathcal{B}_2})$. The diagram is depicted in Figure 2. By exhausting the cylinder with the lines parallel to the line trough the centers of $\mathcal{B}_1$ and $\mathcal{B}_2$, i.e. using a suitable Riemann integral or Fubini's theorem, and applying Theorem 1.(i) we conclude that the volume of $\mathcal{C}_1 \cup \mathcal{C}_2$ is at most the volume of the cylinder $\lambda_{d-1}(B_{d-1}) \cdot \left(\sqrt{\frac{2d}{d+1}}\right)^{d-1}$. $\square$

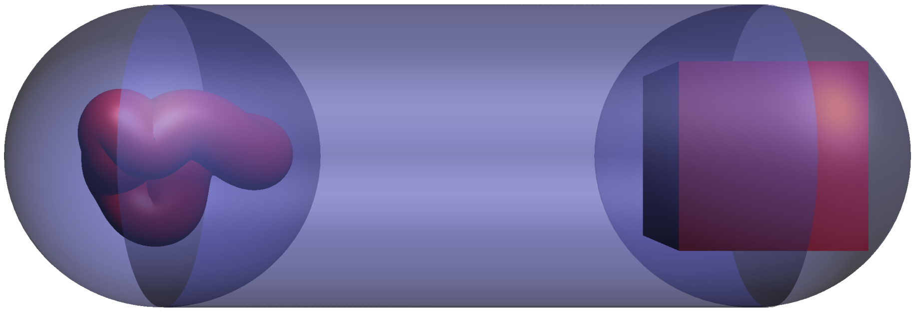

Figure 2: Two 3-dimensional components with the enclosing balls and enclosing cylinder.

The estimates for the first few upper bounds of $l_d(2)$ in Lemma 3 are: $l_2(2) \leq \frac{2}{\sqrt{3}} \approx 1.1547$, $l_3(2) \leq \frac{3\pi}{8} \approx 1.1781$, $l_4(2) \leq \frac{8\sqrt{2}\pi}{15\sqrt{5}} \approx 1.0597$, $l_5(2) \leq \frac{25\pi^2}{288} \approx 0.8567$ and $l_d(2)$ tends to zero as the dimension d approaches infinity.

The enclosing balls are a bit wasteful. The universal cover problem, first stated in a personal communication of Lebesgue in 1914, asks for the minimum area A of a convex set U containing a congruent copy of any planar set of diameter 1, see [8]. For the currently best known bounds $0.832 \leq A \leq 0.844$ and generalizations to higher dimensions we refer the interested reader to [7, Section 11.4]. We will not follow these lines but assume that the connected components are d-dimensional open balls in the first run and provide better upper bounds for the volumes of sets with exactly $n = 2$ such connected components below.

We define $f_d^\circ(n)$ and $l_d^\circ(n)$ by restricting the n connected components to d-dimensional open balls. Clearly we have $f_d^\circ(n) \leq f_d(n)$ and $f_d^\circ(n) \leq l_d^\circ(n) \leq l_d(n)$.

Lemma 4 *For $d, n \in \mathbb{N}$, we have $l_d^\circ(n) \leq \max\left(1, \frac{n}{2^d}\right) \cdot \lambda_d(B_d)$.*

PROOF. Consider n disjoint open d-dimensional balls with diameters $X_1 \leq 1, \cdots, X_n \leq 1$, where we can assume w.l.o.g. that $X_1 \leq \cdots \leq X_n \leq 1$. Clearly in dimension 1 we have $l_1^\circ(n) = l_1(n) = 1 = \lambda_1(B_1)$ for all $n \in \mathbb{N}$ and for all dimensions d, we have $l_d^\circ(1) = l_d(1) = \lambda_d(B_d)$, so in the cases where either d or n is 1 the stated inequality holds, hence we can assume that $d \geq 2$ and $n \geq 2$. By Theorem 1 we have $X_i + X_j \leq 1$ for all $1 \leq i < j \leq n$. If $X_n \leq \frac{1}{2}$ then we have $\sum\limits_{i=1}^{n} X_i^d \leq \frac{n}{2^d}$ so that the stated upper bound is valid. Otherwise we have $X_i \leq 1 - X_n$ and consider the maximization of the function $g_d(x) := x^d + (n-1)(1-x)^d$, where $x \in \left[\frac{1}{2}, 1\right]$. Due to $g_d(x)'' = d(d-1)x^{d-2} + d(d-1)(n-1)(1-x)^{d-2} > 0$ every inner local extremum has to be a minimum so that the global maximum is attained at the boundary of the domain. Finally we compute $g_d(1) = 1$, $g_d\left(\frac{1}{2}\right) = \frac{n}{2^d}$ and conclude the stated upper bound. $\square$

We would like to remark that the special case of balls of diameter $\frac{1}{2}$ is directly related to point sets with pairwise integral distances. Let $\mathcal{P}$ be a d-dimensional open point set consisting of n open d-dimensional balls of diameter $\frac{1}{2}$ without a pair of points at integral distance. Then the distance between

the centers of the balls must be of the form $d_{i,j} + \frac{1}{2}$ for integers $d_{i,j}$. By dilation with a factor of two we obtain a point set where all distance are odd integers. The authors of [23] have shown $n \le d+2$, where equality is possible if and only $if d + 2 \equiv 0 \pmod{16}$, for these point sets. The maximum number of odd integral distances between points in the plane has been exactly determined in [31]. There are similar relations for open balls of diameter $\frac{1}{k}$, where k is an arbitrary integer.

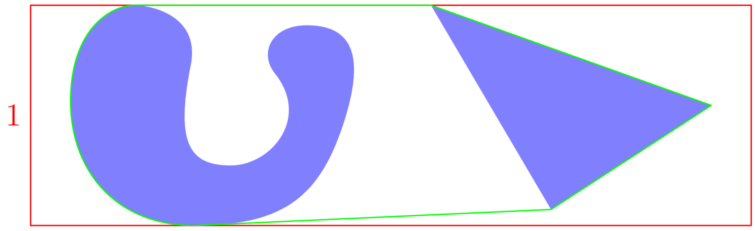

Figure 3: Two components between two parallel lines.

In dimension $d = 2$ the upper bound from Lemma 3 can easily be improved.

Lemma 5

$$l_2(2) \le 1.$$

PROOF. Let $\mathcal{P}$ be a planar open point set consisting of two components $\mathcal{C}_1$ and $\mathcal{C}_2$ each of diameter at most 1. If one of them is contained in the closed convex hull of the other, see Figure 4 for an example, then we have $\lambda_2(\mathcal{P}) \le \lambda_2(B_2) = \frac{\pi}{4} < 1$. Otherwise, we select any support line $\mathcal{L}$ through the boundary points of $\mathcal{C}_1$ and $\mathcal{C}_2$ so that both regions are in the same half-plane determined by $\mathcal{L}$. We then consider the strip parallel to this line with smallest possible width w containing both regions, see Figure 3. Since both $\mathcal{C}_1$ and $\mathcal{C}_2$ have diameter at most 1 we have $w \le 1$. By exhausting the strip with the lines parallel to $\mathcal{L}$ and applying Theorem 1 we conclude that the area of $\mathcal{C}_1 \cup \mathcal{C}_2$ is at most 1. □

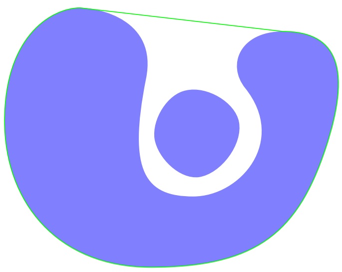

Figure 4: A component contained in the convex hull of another.

By imposing slightly more structure on the sets avoiding integral distances we can find an improved upper bound for their volumes in arbitrary dimension.

Lemma 6 *Let $\mathcal{P}$ be a d-dimensional open point set whose components all have a diameter of at most* 1*. $\mathcal{P}$ contains a pair of points at integral distance if and only if $\Big(\text{dist}(\mathcal{C}_1, \mathcal{C}_2), \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2)\Big) \cap \mathbb{N} \neq \emptyset$ for two of its components.*

PROOF. Since all components are open and have a diameter of at most 1, pairs of points at integral distance need to belong to two different components, say $\mathcal{C}_1$ and $\mathcal{C}_2$. Let $x \in \mathcal{C}_1$, $y \in \mathcal{C}_2$ with $d(x, y) = n \in \mathbb{N}$. We then select two small closed balls $\overline{B}(x, \varepsilon_1) \subsetneq \mathcal{C}_1$ and $\overline{B}(y, \varepsilon_2) \subsetneq \mathcal{C}_2$ centered at x and y, respectively, where $\varepsilon_1, \varepsilon_2 > 0$. The line $\mathcal{L}$ through x and y meets the two balls in intervals $[x_1, x_2] \subsetneq B(x, \varepsilon_1)$ and $[y_1, y_2] \subsetneq B(y, \varepsilon_2)$, where $x_1, x_2 \in \mathcal{C}_1$ and $y_1, y_2 \in \mathcal{C}_2$. With this we have

$$\text{dist}(\mathcal{C}_1, \mathcal{C}_2) < \min_{1 \le i,j \le 2} d(x_i, y_j) < d(x, y) = n < \max_{1 \le i,j \le 2} d(x_i, y_j) < \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2).$$

Conversely, if $\text{dist}(\mathcal{C}_1, \mathcal{C}_2) < n < \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2)$ for an integer n, then there exist $x_1, x_2 \in \mathcal{C}_1$ and $y_1, y_2 \in \mathcal{C}_2$ such that

$$\text{dist}(\mathcal{C}_1, \mathcal{C}_2) < d(x_1, y_1) < n < d(x_2, y_2) < \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2).$$

Joining x_1 with x_2 in $\mathcal{C}_1$ and y_1 with y_2 in $\mathcal{C}_2$ by continuous paths, we can find $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$ on these paths with $d(x, y) = n$. $\square$

We remark that we can reformulate the condition of Lemma 6 as $\text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2) \le \lfloor \text{dist}(\mathcal{C}_1, \mathcal{C}_2) \rfloor + 1$. Next we use Lemma 6 and Proposition 1 to deduce some structural information on the pairs of connected components $\mathcal{C}_1$, $\mathcal{C}_2$ in a d-dimensional integral distance avoiding set $\mathcal{P}$. Due to Lemma 1 there exist parallel hyperplanes $\mathcal{H}_2$ and $\mathcal{H}_3$ such that, after a possible relabeling of the components, $\mathcal{C}_1$ is on the left hand side of $\mathcal{H}_2$, $\mathcal{C}_2$ is on the right hand side of $\mathcal{H}_3$, and $\mathcal{H}_2$ is on the left hand side of $\mathcal{H}_3$. W.l.o.g. we assume $\text{dist}(\mathcal{H}_2, \mathcal{H}_3) \ge \text{dist}(\mathcal{C}_1, \mathcal{C}_2)$. By Lemma 6 there exist two further hyperplanes $\mathcal{H}_1, \mathcal{H}_4$ being parallel to $\mathcal{H}_2$ and $\mathcal{H}_3$ such that $\mathcal{C}_1$ is on the right hand side of $\mathcal{H}_1$ and $\mathcal{C}_2$ is on the left hand side of $\mathcal{H}_4$. In other words, $\mathcal{C}_1$ lies between $\mathcal{H}_1$ and $\mathcal{H}_2$, and $\mathcal{C}_2$ lies between $\mathcal{H}_3$ and $\mathcal{H}_4$. W.l.o.g. we can assume $\text{dist}(\mathcal{H}_1, \mathcal{H}_4) \le \text{diam}(\mathcal{C}_1 \cup \mathcal{C}_2)$. Thus for $d_1 := \text{dist}(\mathcal{H}_1, \mathcal{H}_2)$ and $d_2 := \text{dist}(\mathcal{H}_3, \mathcal{H}_4)$ we have $d_1 + d_2 \le 1$ by Theorem 1.(i). Clearly d_1 and d_2 are upper bounds for the width of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

For a convex body $\mathcal{K}$ in $\mathbb{R}^d$ with diameter D and minimal width ω an upper bound on its d-dimensional volume V has been found in [25, Theorem 1], namely:

$$V \le \lambda_{d-1}(B_{d-1}) \cdot D^d \int_0^{\arcsin \frac{\omega}{D}} \cos^d \theta \, \mathrm{d}\theta. \tag{1}$$

Equality holds if and only if $\mathcal{K}$ is the d-dimensional spherical symmetric slice with diameter D and minimal width ω. In the planar case some more inequalities relating several descriptive parameters of a convex set can be found in [35]. Since we will extensively use d-dimensional spherical symmetric slices with diameter 1 and width $\frac{1}{2}$, we denote them by S_d. Viewing S_d as a truncated d-dimensional ball of unit diameter we denote the two isomorphic cut-off bodies by C_d and call them caps, i.e. we have $\lambda_d(B_d) = \lambda_d(S_d) + 2 \cdot \lambda_d(C_d)$.

$$\lambda_d(S_d) = \lambda_{d-1}(B_{d-1}) \int_0^{\frac{\pi}{6}} \cos^d \theta \, \mathrm{d}\theta, \tag{2}$$

$$\lambda_d(C_d) = \frac{1}{2} \cdot (\lambda_d(B_d) - \lambda_d(S_d)). \tag{3}$$

In Table 1 we give the first exact volumes and refer to the appendix for further equivalent expressions.

$\mathbf{d}$	2	3	4	5
$\lambda_{\mathbf{d}}(\mathbf{S}_{\mathbf{d}})$	$\frac{\sqrt{3}}{8} + \frac{\pi}{12} \approx 0.4783$	$\frac{11\pi}{96} \approx 0.3600$	$\frac{\pi}{384} \cdot \left(9\sqrt{3} + 4\pi\right) \approx 0.2303$	$\frac{203\pi^2}{15360} \approx 0.1304$
$\lambda_{\mathbf{d}}(\mathbf{C}_{\mathbf{d}})$	$\frac{\pi}{12} - \frac{\sqrt{3}}{16} \approx 0.1535$	$\frac{5\pi}{192} \approx 0.0818$	$\frac{\pi^2}{96} - \frac{3\sqrt{3}\pi}{256} \approx 0.0390$	$\frac{53\pi^2}{30720} \approx 0.0170$

Table 1: Values of $\lambda_d(S_d)$ and $\lambda_d(C_d)$ for small dimensions.

Lemma 7 *For $d \ge 2$, we have $f_d(2) \le 2\lambda_d(S_d)$.*

PROOF. With notation introduced above we estimate the total volume of the closed convex hulls of the two connected components $\text{conv}(\overline{\mathcal{C}}_1), \text{conv}(\overline{\mathcal{C}}_2)$,

$$\lambda_d(\text{conv}(\mathcal{C}_1)) + \lambda_d(\text{conv}(\mathcal{C}_2)),$$

where both components have a diameter of at most 1, $\mathcal{C}_1$ has a width of at most d_1, and $\mathcal{C}_2$ has a width of at most d_2, using Inequality (1). We thus have

$$\lambda_d(\mathrm{conv}(\mathcal{C}_1)) \le \lambda_d(B_{d-1}) \int_0^{\arcsin d_1} \cos^d \theta \, \mathrm{d}\theta$$

and

$$\lambda_d(\mathrm{conv}(\mathcal{C}_2)) \le \lambda_d(B_{d-1}) \int_0^{\arcsin d_2} \cos^d \theta \, \mathrm{d}\theta.$$

Since both right hand sides are strictly monotone in d_1, d_2, respectively, we can assume w.l.o.g. that $d_1 + d_2 = 1$. So it suffices to maximize the function

$$\int_0^{\arcsin x} \cos^d \theta \, \mathrm{d}\theta + \int_0^{\arcsin(1-x)} \cos^d \theta \, \mathrm{d}\theta$$

for $x \in [0, 1]$. After a straightforward calculation we conclude that the unique maximum is attained at $x = \frac{1}{2}$. □

One might conjecture that this upper bound is also valid for $l_d(2)$, see Conjecture 1. In this context we would like to remark that related problems can be quite complicated, e.g. it is quite hard to determine the equilateral n-gon with diameter 1 and maximum area [3, 4].

Using a simple averaging argument we can extend each upper bound for n components to each larger number of components in the same dimension.

Lemma 8 *If $l_d(n) \le \Lambda$ then we have $l_d(k) \le \frac{k}{n} \cdot \Lambda$ for all $k \ge n$.*

PROOF. Let $\mathcal{P}$ be a d-dimensional open point set consisting of k connectedcomponents. The volume of each of the $\binom{k}{n}$ different subsets consisting of n connected components is at most Λ. Since each component occurs exactly $\binom{k-1}{n-1}$ times in those sets the stated upper bound follows. □

Clearly the same argument applies to $f_d(n)$, $l_d^\circ(n)$, and $f_d^\circ(n)$.

3 Constructions

In dimension one we can consider one open interval of length $1 - \varepsilon$ and $n - 1$ open intervals of length $\frac{\varepsilon}{n}$, where $1 > \varepsilon > 0$, arranged in the unit interval so that they are pairwise non-intersecting. Obviously there is no pair of points at integral distance and the total length of the n intervals tends to 1 as ε approaches zero. Thus we can conclude $f_1(n) = l_1(n) = f_1^\circ(n) = l_1^\circ(n) = 1$ from Theorem 1. For $n = 1$ component the unique example achieving the maximum volume of $f_d(1) = l_d(1) = f_d^\circ(1) = l_d^\circ(1) = \lambda_d(B_d)$ is the d-dimensional open ball with diameter one. If all components are open d-dimensional balls then the upper bound of Lemma 4 can be indeed achieved for dimensions $d \ge 2$.

Theorem 2 *For $d \ge 2$ we have $l_d^\circ(n) = \max\left(1, \frac{n}{2^d}\right) \cdot \lambda_d(B_d)$.*

PROOF. Due to Lemma 4 it remains to provide constructions (asymptotically) achieving the upper bound.

For an arbitrary $1 > \varepsilon > 0$ we consider one d-dimensional open ball of diameter $1 - \varepsilon$ and $n - 1$ open balls of diameter $\frac{\varepsilon}{n-1}$ arranged in the interior of an open ball $\mathcal{B}$ of diameter 1 so that they do not intersect. As all components are contained in $\mathcal{B}$ there are no two points at integral distance. As ε tends to zero the volume of the n balls tends to $\lambda_d(B_d)$.

For the remaining part we consider n open d-dimensional balls with diameter $\frac{1}{2}$ and centers located at the corners of a regular n-gon with radius k of its circumcircle. If k is large enough then there is no line intersecting three or more balls. □

Another construction consists of n open d-dimensional balls with centers $\left(i \cdot k, i^2, 0, \ldots, 0\right)$ and diameter $\frac{1}{2}$ for $1 \le i \le n$. If k is large enough then again there is no line intersecting three or more balls.

Corollary 1 *For $d \geq 2$ and $n \leq 2^d$, we have $f_d^\circ(n) = l_d^\circ(n) = \lambda_d(B_d)$.*

It turns out that in fact the equalities $f_d^\circ(n) = l_d^\circ(n) = \max\left(1, \frac{n}{2^d}\right) \cdot \lambda_d(B_d)$ hold for all dimensions $d \geq 2$. To explain the underlying idea, we first consider the special case where $d = 2$ and $n = 5$, i.e. the first case which is not covered by Corollary 1.

Proposition 2

$$f_2^\circ(5) = \frac{5\pi}{16} \approx 0.9817477.$$

PROOF. For each integer $k \geq 2$ and $\frac{1}{7} > \varepsilon > 0$, we consider a regular pentagon P with side length $\frac{1}{2} - 2\varepsilon + k$. At each of the corners of P we place an open circle of diameter $\frac{1}{2} - 2\varepsilon$, see Figure 5. Since each component has a diameter less than 1 there is no pair of points at integral distance inside one of the five components. For two points a and b from different components we either have

$$k < \operatorname{dist}(a, b) < k + 1,$$

whenever the discs are adjacent with their centers on an edge of P, or or

$$\left(\frac{1+\sqrt{5}}{2}\right) \cdot k + \frac{\sqrt{5}-1}{4} - 2\varepsilon < \operatorname{dist}(a, b) < \left(\frac{1+\sqrt{5}}{2}\right) \cdot k + \frac{3+\sqrt{5}}{4} - 5\varepsilon$$

otherwise.

Let $[\alpha]$ denote the positive fractional part of a real number α, i.e. there exists an integer l with $\alpha = l + [\alpha]$ and $0 \leq \alpha < 1$. If, for a given $\varepsilon > 0$, we can find an integer k such that $\left[\left(\frac{1+\sqrt{5}}{2}\right) \cdot k + \frac{\sqrt{5}-1}{4} - 2\varepsilon\right] < 3\varepsilon$, then the point set with parameters k and ε does not contain a pair of points at integral distance.

Since $\frac{1+\sqrt{5}}{2}$ is irrational we can apply the equidistribution theorem, see e.g. [37, 39], to conclude that $\left(\frac{1+\sqrt{5}}{2}\right) \cdot \mathbb{N}$ is dense (even uniformly distributed) in $[0, 1)$. The same is true if we add a fixed real number $\frac{\sqrt{5}-1}{4} - 2\varepsilon > 0$. Thus we can find a suitable integer k for each $\varepsilon > 0$. As ε tends to zero the total area of the five components approaches $\frac{5\pi}{16}$ which is best possible by Lemma 4. $\square$

We would like to demonstrate this by a short list of suitable multipliers k: $\left[\frac{\sqrt{5}-1}{4} + \left(\frac{1+\sqrt{5}}{2}\right) \cdot \mathbf{6}\right] \approx 0.01722$, $\left[\frac{\sqrt{5}-1}{4} + \left(\frac{1+\sqrt{5}}{2}\right) \cdot \mathbf{61}\right] \approx 0.00909$, $\left[\frac{\sqrt{5}-1}{4} + \left(\frac{1+\sqrt{5}}{2}\right) \cdot \mathbf{116}\right] \approx 0.00096$, $\left[\frac{\sqrt{5}-1}{4} + \left(\frac{1+\sqrt{5}}{2}\right) \cdot \mathbf{1103}\right] \approx 0.00051$, and $\left[\frac{\sqrt{5}-1}{4} + \left(\frac{1+\sqrt{5}}{2}\right) \cdot \mathbf{2090}\right] \approx 0.00005$.

We shall generalize Proposition 2 to an arbitrary dimension $d \geq 2$ and arbitrary number n of components. The idea is to locate n small d-dimensional balls of diameter slightly less than $\frac{1}{2}$ at points C_i in a two-dimensional sub-plane in such a way that the set of different pairwise distances α_i between the centers are linearly independent over the rational numbers. So either the distances coincide or they are rationally independent. A quite natural candidate for the center points C_i are the corners of a regular p-gon, where p is an odd prime. We will use a theorem of Mann, see [29], to prove our assumption on the distances. The condition that the point set avoids integral distances can be translated into a system of the form $[\alpha_1 \cdot k] < \varepsilon, \ldots, [\alpha_l \cdot k] < \varepsilon$, where $k \in \mathbb{N}$, i.e. we want an integer k such that the fractional part of the scaled pairwise distances are arbitrarily small. By a theorem of Weyl, see e.g. [39], those systems have solutions if the α_i are linearly independent over $\mathbb{Q}$.

As a motivation, we would like to remark that some independence of the α_i over the integers is indispensable, e.g. a similar construction using the corners of a regular hexagon does not work, since the lengths of the occurring diagonals are given by 1, $\sqrt{3}$, and 2. We remark that very recently Mann's theorem was used in another problem from discrete geometry, see [14, 34].

Theorem 3 *(Mann, 1965, [29]) Suppose we have*

$$\sum_{i=1}^{k} a_i \zeta_i = 0,$$

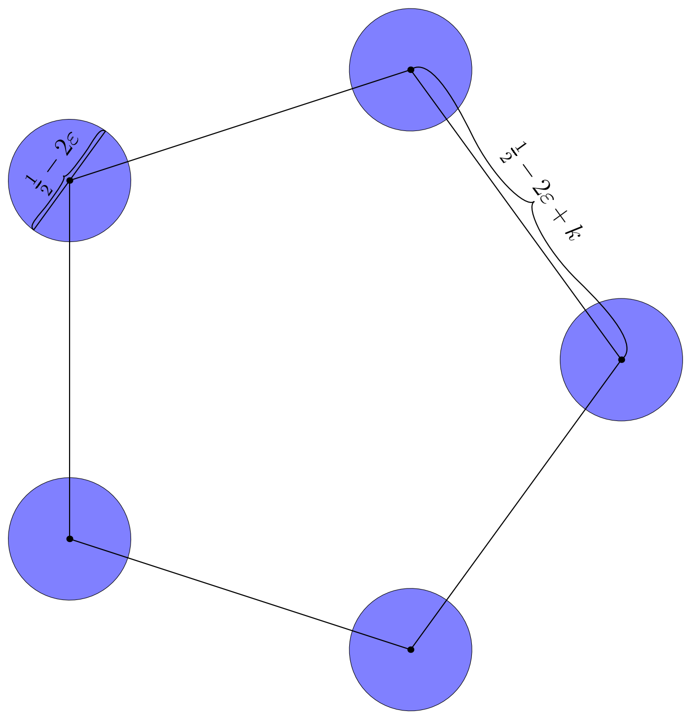

Figure 5: Integral distance avoiding point set for $d = 2$ and $n = 5$.

with $a_i \in \mathbb{Q}$, the ζ_i roots of unity, and no sub-relations $\sum\limits_{i\in I} a_i\zeta_i = 0$, where $\emptyset \neq I \subsetneq [k]$. Then

$$(\zeta_i/\zeta_j)^m = 1$$

for all i, j, with $m = \prod\limits_{p\le k \text{ prime}} p$.

The corner points of a regular p-gon with a circumcircle of radius 1 centered at the origin can be given as

$$\left(\cos\left(\frac{j\cdot 2\pi}{p}\right), \sin\left(\frac{j\cdot 2\pi}{p}\right)\right)$$

for $0 \le j \le p-1$. Using complex number they coincide with the p-th roots of unity $\zeta_j' = \cos\left(\frac{j\cdot 2\pi}{p}\right) + i\cdot\sin\left(\frac{j\cdot 2\pi}{p}\right)$. The distance between corner 0 and corner j is given by $2\sin\left(\frac{j\cdot 2\pi}{2p}\right)$. Since $\sin(\pi-\alpha) = \sin(\pi)$, there are only $(p-1)/2$ distinct distances in a regular p-gon, attained for $1 \le j \le (p-1)/2$. We remark that this is not too far away from the minimal number of distinct distances in the plane, which is given by $c\cdot\frac{p}{\log p}$ for a suitable constant c, see [24]. We can express these distances in terms of $2p$-th roots of unity $\zeta_j = \cos\left(\frac{j\cdot 2\pi}{2p}\right) + i\cdot\sin\left(\frac{j\cdot 2\pi}{2p}\right)$ via

$$2\sin\left(\frac{j\cdot 2\pi}{2p}\right) = \frac{\zeta_j - \zeta_{2p-j}}{i}$$

for all $1 \le j \le \frac{p-1}{2}$.

Lemma 9 *For a given odd prime number p let $\alpha_j = \frac{\zeta_j - \zeta_{2p-j}}{i}$ for $1 \le j \le \frac{p-1}{2}$, where the ζ_j are $2p$-th roots of unity. Then the α_j are linearly independent over $\mathbb{Q}$.*

PROOF. Suppose to the contrary that there are rational numbers b_j for $1 \le j \le l \le \frac{p-1}{2}$ with $\sum\limits_{j=1}^{l} b_j\alpha_j = 0$. Then we have

$$\sum_{j=1}^{l} (b_j\zeta_j - b_j\zeta_{2p-j}) = 0.$$

Now let J be a subset of those indices j, $2p-j$ such that $\sum_{j\in J} a_j\zeta_j = 0$, where $a_j \in \{\pm b_j\}$, and no sub-relation adds up to zero. We have $|J| \le p-1$. Thus we conclude from Mann's Theorem $(\zeta_{j_1}/\zeta_{j_2})^2 = 1$ for all $j_1, j_2 \in J$ since

$$\gcd\Big(2p, \prod_{t\le p-1 \text{ prime}} t\Big) = 2.$$

With this we obtain $j_2 = j_1 + p$ for $j_1 < j_2$. Since J is a subset of

$$\left\{1,\dots,\frac{p-1}{2}\right\} \cup \left\{2p-\frac{p-1}{2},\dots,2p-1\right\},$$

this is not possible and the α_j have to be linearly independent over $\mathbb{Q}$. $\square$

Theorem 4 *For $d \ge 2$ we have $f_d^\circ(n) = \max\big(1, \frac{n}{2^d}\big) \cdot \lambda_d(B_d)$.*

PROOF. Since $f_d^\circ(n) \le l_d^\circ(n)$ we conclude from Theorem 2 the upper bound $f_d^\circ(n) \le \max\big(1, \frac{n}{2^d}\big) \cdot \lambda_d(B_d)$. For the construction we fix an odd prime p with $p \ge n$. For each integer $k \ge 2$ and each $\frac{1}{4} > \varepsilon > 0$ we consider a regular p-gon P with side lengths $2k \cdot \sin\left(\frac{\pi}{p}\right)$. At n arbitrarily chosen corners of the p-gon P we place the centers of d-dimensional open balls with diameter $\frac{1}{2} - 2\varepsilon$. Since each of the n components has a diameter less than 1 there is no pair of points inside one of these n components. Next we consider two points a and b from two different components. By α we denote the distance of the centers of the corresponding open balls. From the triangle inequality we conclude

$$\alpha - \left(\frac{1-4\varepsilon}{2}\right) < \text{dist}(a,b) < \alpha + \left(\frac{1-4\varepsilon}{2}\right).$$

Since the occurring distances α are given by $2k\sin\left(\frac{j\pi}{p}\right)$ for $1 \le j \le \frac{p-1}{2}$ we look for a simultaneous solution of the system

$$\left[2k \cdot \sin\left(\frac{j\pi}{p}\right) - \frac{1}{2} + 2\varepsilon\right] \le 4\varepsilon$$

with $k \in \mathbb{N}$. By Lemma 9 the factors $2\sin\left(\frac{j\pi}{p}\right)$ are linearly independent over $\mathbb{Q}$, so by Weyl's Theorem [39] the systems are consistent for all k.

Therefore, for every $0 < \varepsilon < \frac{1}{4}$ we can choose a suitable value of k and construct a point set without pairs of points at integral distances with a volume of $n\left(\frac{1}{2} - 2\varepsilon\right)^d \cdot \lambda_d(B_d)$. As ε approaches zero this volume tends to $\frac{n}{2^d} \cdot \lambda_d(B_d)$. For small values of n we consider, for an arbitrary $\varepsilon > 0$, one open d-dimensional ball of diameter $1-\varepsilon$ and $d-1$ open d-dimensional balls of diameter $\frac{\varepsilon}{d}$ arranged in an open d-dimensional ball of diameter 1, see the proof of Theorem 2. As the diameter of the whole set is less than 1 there is no pair of points at integral distance. $\square$

Thus, in the case of spherical components the values of $l_d^\circ(n)$ and $f_d^\circ(n)$ have been completely determined. For general components the situation is more challenging for $n \ge 2$; if $n = 1$ then the extremal component is indeed a d-dimensional open ball. The general construction idea that turns out to be successful is to consider d-dimensional open balls of diameter larger than $\frac{1}{2}$, which are truncated in several directions. A planar construction of two such truncated circles is depicted in Figure 6.

Lemma 10 *For $n \ge 2$ we have $l_d(n) \ge n \cdot \lambda_d(S_d)$.*

PROOF. Consider n truncated d-dimensional balls S_d with centers located at $\big(i\cdot k, i^2\cdot k, 0, \dots, 0\big)$ and diameter 1 for $1 \le i \le n$. If k is large then there is no line intersecting three or more components. $\square$

From a remote stand point, the centers of the truncated balls seem to be located on the same line. We conjecture this construction to be tight in general, compare Lemma 7.

Conjecture 1 *For $n \ge 2$ and $d \ge 2$ we have $l_d(n) = n \cdot \lambda_d(S_d)$.*

For $n \geq 3$, the configuration in Lemma 10 contains pairs of points at integral distance, since in $\mathbb{R}$ there cannot be three points with pairwise odd integral distances. But at least for $n = 2$ components the construction also works for integral distance avoiding sets.

Lemma 11

$$f_d(2) \geq 2 \cdot \lambda_d(S_d).$$

PROOF. For an arbitrary integer $k \geq 5$ we place a d-dimensional ball with diameter $1 - \frac{2}{k}$ at the origin and cut off the spherical cap at the hyperplanes with value $\pm\left(\frac{1}{4} - \frac{1}{k}\right)$ of the first coordinate. By $\mathcal{S}_1$ we denote the arising truncated ball. Another such truncated ball $\mathcal{S}_2$ is located with a shift of $dk + \frac{1}{2} - \frac{2}{k}$ in the direction of the first coordinate. (See Figure 6 for a drawing of the two-dimensional case.) Both $\mathcal{S}_1$ and $\mathcal{S}_2$ have a diameter less than 1 for all $k \in \mathbb{N}$ so that they contain no pair of points at integral distance. For two points $a \in \mathcal{S}_1$ and $b \in \mathcal{S}_2$ we have

$$dk < \operatorname{dist}(a, b) < \sqrt{(d-1)\left(1 - \frac{2}{k}\right)^2 + \left(dk + 1 - \frac{4}{k}\right)^2} \leq dk + 1,$$

so that $\mathcal{S}_1 \cup \mathcal{S}_2$ contains no pair of points at integral distance.

Finally we remark that the volume of $\mathcal{S}_1 \cup \mathcal{S}_2$ approaches $2 \cdot \lambda_d(S_d)$ as k tends to ∞. $\square$

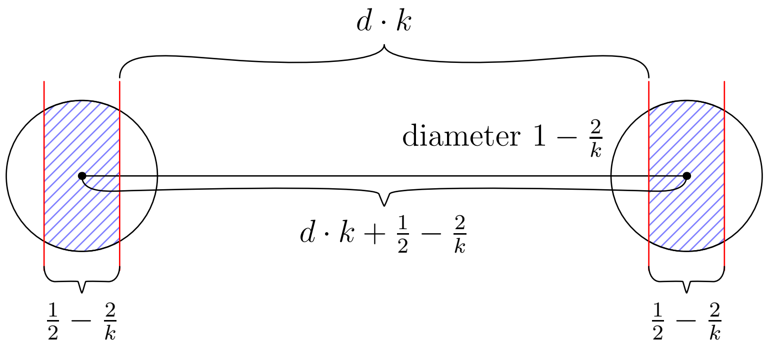

Figure 6: Truncated circles – a construction of two components without integral distances.

Combining Lemma 7 with Lemma 11 yields:

Theorem 5 *For $d \geq 2$ we have $f_d(2) = 2\lambda_d(S_d)$.*

For dimensions $2 \leq d \leq 4$ and $n \leq 2^d$ we provide another construction:

Lemma 12 *(Cube construction) For dimension $2 \leq d \leq 4$ we have $f_d(2^d) \geq 1$.*

PROOF. For each integer $k \geq \max(d, 5) = 5$ we consider the following d-dimensional point set consisting of 2^d open cubes of widths $\frac{1}{2} - \frac{2}{k}$ whose centers are located at the corners of a d-dimensional box and whose edges are parallel to the standard coordinate axes. As label for such a corner (or center of a cube) we use vectors $v = (v_1, \ldots, v_d) \in \{0, 1\}^d$. With this the coordinates are given by

$$\left(v_d \cdot \left(k^{4^1} + \frac{1}{2}\right), \ldots, v_{d+1-i} \cdot \left(k^{4^i} + \frac{1}{2}\right), \ldots, v_1 \cdot \left(k^{4^d} + \frac{1}{2}\right)\right).$$

Since the diameter of the cubes $\frac{\sqrt{d}}{2}$ is at most 1 for $d \leq 4$ every open cube is free of pairs of points at integral distance. It remains to check the distances between pairs of points in different cubes. Due to symmetry it suffices to consider the distance between a point a in the cube corresponding to the label $(0, \ldots, 0)$ and a point b in the cube corresponding to the label $v = (v_1, \ldots, v_d) \neq (0, \ldots, 0)$. Let j be

the minimal index such that $v_j = 1$. As abbreviation we use $h := d + 1 - j$. Applying Pythagoras' theorem for two farthest vertices of the two cubes we conclude $k^{4^h} < \mathrm{dist}(a, b)$

$$
\begin{aligned}
&\le \sqrt{\sum_{i=1}^{h} \underbrace{\left(1-\frac{2}{k}+k^{4^i}\right)}_{\ge \frac{1}{2}-\frac{2}{k}}^{2} + (d-h)\cdot\left(\frac{1}{2}-\frac{2}{k}\right)^{2}} \\
&\le \sqrt{\sum_{i=1}^{h} \left(1-4k^{4^i-1}+2k^{4^i}+k^{2\cdot 4^i}\right)+\frac{d}{4}} \\
&\le \sqrt{h-4k^{4^h-1}+2k^{4^h}+k^{2\cdot 4^h}+3(h-1)k^{2\cdot 4^{h-1}}+\frac{d}{4}} \\
&\le \sqrt{2k^{4^h}+k^{2\cdot 4^h}-4\left(k^{4\cdot 4^{h-1}-1}-k^{2\cdot 4^{h-1}+1}\right)} \\
&< k^{4^h}+1.
\end{aligned}
$$

Thus there is no pair of points at integral distance. As k tends to infinity the volume of each cube approaches $\frac{1}{2^d}$. A two-dimensional example is depicted in Figure 7, were we remark that the usual x-axis corresponds to the second coordinate and we have drawn circles of diameter one around the squares in anticipation of the shortly following more general *box construction*. $\square$

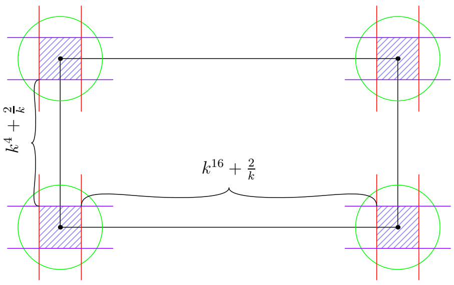

Figure 7: Cube construction for dimension $d = 2$.

By considering subsets of the cubes of the previous construction we can obtain a lower bound for all $n \le 2^d$.

Corollary 2 *For $2 \le d \le 4$ and $n \le 2^d$ we have $f_d(n) \ge \frac{n}{2^d}$.*

We remark that this bound can be improved slightly whenever the number of components is small. For example if we consider four 3-dimensional cubes whose centers are located in a common hyperplane $\mathcal{H}$ then the cubes can be enlarged in the direction perpendicular to $\mathcal{H}$ while preserving the property of avoiding integral distances, see Figure 8.

For general dimension $d \ge 2$ and arbitrary $n \le 2^d$ we propose the following improved construction, which we call the *box-construction*, based on the key idea of the proof of Lemma 12. We choose the n lexicographically minimal binary vectors $v \in \{0,1\}^d$ as labels of the corners (corresponding to centers of components) of the *growing* large box. To start we place at each corner a d-dimensional ball of diameter $1 - \frac{2}{k}$. Later on we will truncate these balls, so that the components of our final point set will be subsets of these balls. Let us label the standard coordinate axes from a_1 to a_d. For each ball $\mathcal{B}$ with label $(v_1, \ldots, v_d)$ and each index $1 \le i \le d$ we cut off the spherical caps in direction of coordinate axis a_{d+1-i} by two hyperplanes being perpendicular to a_{d+1-i} at distance $\frac{1}{4} - \frac{1}{k}$ to the center of $\mathcal{B}$ if $(v_1, \ldots, v_{i-1}, 1 - v_i, 0, \ldots, 0)$ is the label of a used ball.

In Figure 9 we have depicted the situation for dimension $d = 2$ and $n = 3$ components. (Again, the usual x-axis corresponds to the second coordinate.) Here the labels of centers are given by $L =$

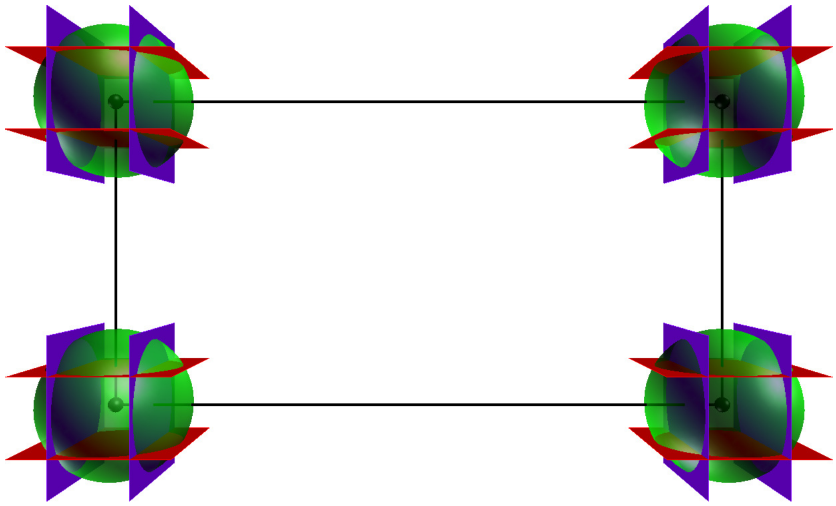

Figure 8: Box construction: Integral distance avoiding set for $d=3$ and $n=4$.

$\Big\{(0,0),(0,1),(1,0)\Big\}$. Since both $(0,1)\in L$ and $(1,0)\in L$ the ball coresponding to $(0,0)$ is cut off in direction of coordinate axis a_1 and a_2. Similarly the ball corresponding to $(0,1)$ is cut off in direction of coordinate axis a_1 and a_2 since $(0,0)\in L$ and $(1,0)\in L$. Since $(1,1)\notin L$ the ball corresponding to $(1,0)$ is not cut of in direction of coordinate axis a_1 but only in direction of coordinate axis a_2 (since $(0,0)\in L$). The parameterized coordinates are given by $(0,0)$, $\left(k^{4^1}+\frac{1}{2},0\right)$, and $\left(0,k^{4^2}+\frac{1}{2}\right)$. If k tends to infinity we obtain the lower bound of $f_2(3)\geq\frac{\pi}{12}+\frac{\sqrt{3}}{8}+\frac{1}{2}\approx 0.9783057$.

Lemma 13 *For $d\geq 2$, $k\geq\max(d,5)$, and $n\leq 2^d$ the previously described box construction results in a d-dimensional integral distance avoiding point set consisting of n components.*

PROOF. Let $(u_1,\dots,u_d)$, $(v_1,\dots,v_d)$ be two labels of used components and j be the first index where they differ, i.e. $u_i=v_i$ for all $i<j$ and $u_j\neq v_j$. As abbreviation we set $h:=d+1-j$. Let a be a arbitrary point of the first and b be an arbitrary point of the second component. From the construction we conclude $\text{dist}(a,b)>k^{4^h}$ and that both balls are cut off in the direction of coordinate axis a_h. Thus we have

$$\text{dist}(a.b)\leq\sqrt{\left(1-\frac{2}{k}+k^{4^h}\right)^2+\sum_{i=1}^{h-1}\left(\frac{3}{2}-\frac{2}{k}+k^{4^i}\right)^2+(d-h)\cdot\left(1-\frac{2}{k}\right)^2},$$

where we assume that a and b even may be contained in the smallest circumscribed box for each component. A similar estimation as in the proof of Lemma 12 yields $\text{dist}(a,b)<k^{4^h}+1$. $\square$

We would like to remark that for $n=2$ the box construction coincides with the one of Lemma 11. In order to state the resulting asymptotic volumes we denote by S_d^k the d-dimensional open ball of unit diameter which is cut off in the directions of the first k coordinate axes with width $\frac{1}{2}$, where $0\leq k\leq d$. For some parameters of k the corresponding volumes can be stated easily, e.g. we have $\lambda_d(S_d^0)=\lambda_d(B_d)$, $\lambda_d(S_d^1)=\lambda_d(S_d)=\lambda_d(B_d)-2\lambda_d(C_d)$, $\lambda_d(S_d^d)=\frac{1}{2^d}$ for $d\leq 4$, and

$$\begin{aligned}\lambda_3(S_3^2) &= 2\int\limits_{-\frac{1}{4}}^{\frac{1}{4}}\int\limits_{-\frac{1}{4}}^{\frac{1}{4}}\sqrt{\left(\frac{1}{2}\right)^2-x^2-y^2}\,\mathrm{d}\,x\,\mathrm{d}\,y\\ &= \frac{\sqrt{2}}{24}+\frac{11}{48}\arcsin\left(\frac{1}{\sqrt{3}}\right)+\frac{19}{48}\arctan\left(\frac{1}{\sqrt{2}}\right)-\frac{1}{6}\arctan\left(\frac{5}{\sqrt{2}}\right)\\ &\approx 0.2277416.\end{aligned}$$

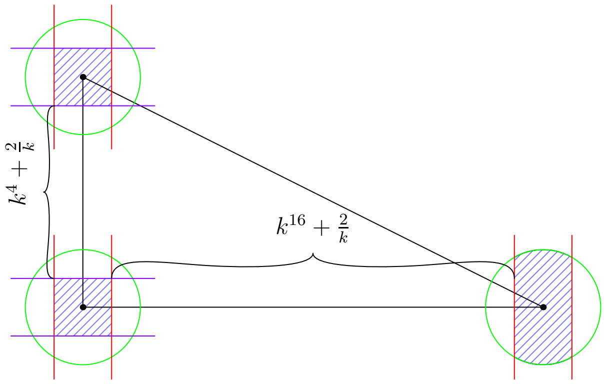

Figure 9: Box construction: Integral distance avoiding point set for $d = 2$ and $n = 3$.

Corollary 3 *For $d \geq 2$ and $1 \leq n \leq 2^d$ we have $f_d(n) \geq h(d, n, 0)$, where $h(d, n, \beta)$ is recursively defined as follows. To this end let $n = 2^k + l$ with $0 \leq l < 2^k$.*

$$h(d,n,\beta) = h(d, 2^k + l, \beta) = \begin{cases} 0 & \text{if } n = 0, \\ 2^k \cdot \lambda_d\left(S_d^{\lceil \log_2 n \rceil + \beta}\right) + h(d, l, \beta + 1) & \text{if } n \geq 1. \end{cases}$$

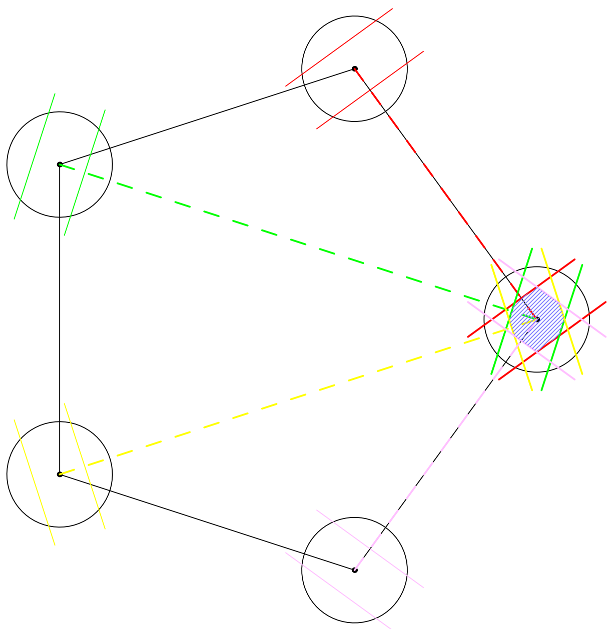

Figure 10: p-gon construction: Integral distance avoiding point set for $d = 2$ and $p = n = 5$.

For general parameters d and n we can improve slightly the construction from Theorem 4. For $d \geq 2$ we choose a prime $p \geq n$ and locate n open balls of diameter $1 - 2\varepsilon$, where ε is suitably chosen, at n out of the p corner points of the regular p-gon. For each two balls we cut off spherical caps in the directions of the lines connecting two centers resulting in a width of $\frac{1}{2} - 2\varepsilon$. If the radius of the regular p-gon approaches infinity we can assume that ε can tend to zero. So to compute the asymptotic volume of this construction it suffices to consider a regular p-gon P of fixed radius > 2, where we locate n open balls at the corners of P and cut off spherical caps such that the components have a width of $\frac{1}{2}$ in the direction of each line connecting two used corner points, i.e. the centers of the n balls. For future reference we call this construction the *p-gon construction*. An example with $p = n = 5$ in dimension $d = 2$ is depicted in figures 10 and 11. The four different cuts for one of the five circles are visualized in Figure 10. The resulting blue octagon, which is slightly larger then a circle of diameter $\frac{1}{2}$, is enlarged

in Figure 11. In the appendix we compute the asymptotic area of this construction from which we can conclude $f_2(5) \geq 1.0633$.

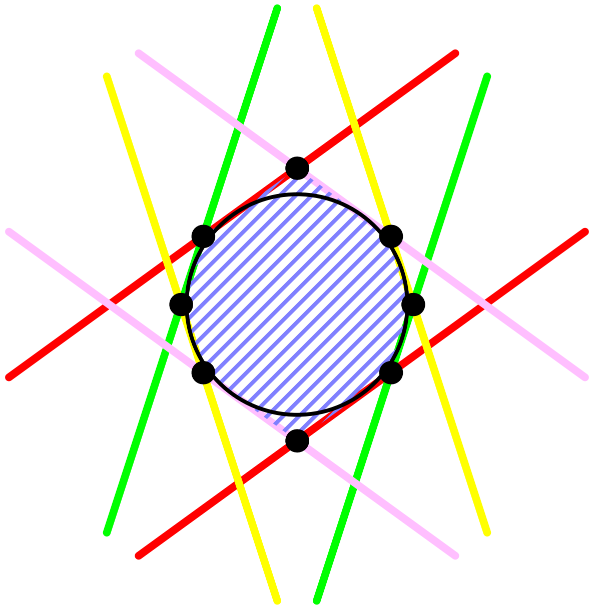

Figure 11: 5-gon construction: One of the resulting octagons.

4 Conclusion

Problems related to point sets with pairwise rational or integral distances were one of Erdős' favorite subjects in combinatorial geometry. In the present paper we study some kind of an inverse problem by asking for the largest open d-dimensional set $\mathcal{P}$ of points without a pair of points at integral distance, i.e. that with the largest possible volume $f_d(n)$, where n denotes the number of connected components of $\mathcal{P}$. As a relaxation we have also considered d-dimensional open point sets consisting of n connected components, each having a diameter at most 1, such that the intersection with every line has a total length of at most one. The corresponding maximal volume was denoted by $l_d(n)$. While the assumption on diameters of the connected components seems to be a bit technical, geometrical objects whose intersections with lines or higher-dimensional subspaces are specified, should be interesting in their own right. So far we were not able to prove that the maximal volume is bounded if we drop the condition on the diameters. In this context we just mention the famous Kakeya problem, i.e. there exists a plane set of area zero containing a unit line segment in every direction, see e.g. [12, Problem G6].

By restricting the shapes of the connected components to d-dimensional balls, we were able to determine the exact values of the corresponding maximal volumes $f_d^\circ(n)$ and $l_d^\circ(n)$, respectively. For $l_d(n)$ we have given a construction which we believe is tight. The determination of $f_d(n)$ is widely open and, except for few exact values, we could only give lower and upper bounds. A summary of what we know for small parameters d and n is provided in Table 2. The upper bounds are based on Lemma 7 and Lemma 8. For the *box construction* see Corollary 3 and for the *p-gon construction* see Theorem 4 and Figure 10. We observe that the box construction is not monotone in the number n of components for dimensions $d \geq 3$, but of course we have $f_d(n+1) \geq f_d(n)$ for all integers d and n.

We would like to see further research in the direction of the proposed problem, e.g. we ask for better constructive lower bounds for $f_d(n)$, when n is large, i.e. to beat the p-gon constructions.

value	lower bound	construction	upper bound
$f_2(3)$	$\frac{\pi}{12}+\frac{\sqrt{3}}{8}+\frac{1}{2}\approx 0.9781$	box	$\frac{3\sqrt{3}}{8}+\frac{\pi}{4}\approx 1.4349$
$f_2(4)$	1	box	$\frac{\sqrt{3}}{2}+\frac{\pi}{3}\approx 1.9132$
$f_2(5)$	≈ 1.0633	5-gon	$\frac{5\sqrt{3}}{8}+\frac{5\pi}{12}\approx 2.3915$
$f_2(7)$	≈ 1.4199	7-gon	$\frac{7\sqrt{3}}{8}+\frac{7\pi}{12}\approx 3.3481$
$f_3(3)$	$2\lambda_3(S_3^2)+\lambda_3(S_3^1)\approx 0.8155$	box	$\frac{11\pi}{32}\approx 1.0799$
$f_3(4)$	$4\lambda_3(S_3^2)\approx 0.9110$	box	$\frac{11\pi}{24}\approx 1.4399$
$f_3(5)$	$\frac{1}{2}+\frac{11\pi}{96}\approx 0.8600$	box	$\frac{55\pi}{96}\approx 1.7999$
$f_3(6)$	$4\lambda_3(S_3^3)+2\lambda_3(S_3^2)\approx 0.9555$	box	$\frac{11\pi}{16}\approx 2.1598$
$f_3(7)$	$\frac{7}{8}=0.8750$	box	$\frac{77\pi}{96}\approx 2.5198$
$f_3(8)$	1	box	$\frac{11\pi}{12}\approx 2.8798$
$f_3(5)$	≈ 0.9867	5-gon	$\frac{55\pi}{96}\approx 1.7999$
$f_3(7)$	≈ 1.3237	7-gon	$\frac{77\pi}{96}\approx 2.5198$

Table 2: Lower and upper bounds for $f_d(n)$.

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A Proofs and additional calculations

PROOF OF CONSTRUCTION 1. First we remark that both $\mathcal{A}_n^d$ and $\mathcal{A}_{n+1}^d$ meet $\mathcal{B}_n^d$ for $n \ge 1$. Thus $\mathcal{P}$ is a connected open set in $\mathbb{R}^d$. The volume $\lambda_d\left(\mathcal{A}_n^d\right)$ is given by

$$\lambda_d(B_d)\cdot\left(\left(2n+\frac{2}{dn^d}\right)^d-(2n)^d\right)=\lambda_d(B_d)\cdot 2^d\cdot\left(\left(n+\frac{1}{dn^d}\right)^d-n^d\right)\ge\lambda_d(B_d)\cdot 2^d\cdot\frac{1}{n}.$$

Since the harmonic series diverges to infinity, the d-dimensional volume of $\mathcal{P}$ is unbounded.

Now we consider the intersection of a line $\mathcal{L}$ with an d-dimensional annulus $\mathcal{C}_d(r_1, r_2)$ with inner radius r_1 and outer radius r_2 centered at the origin. Due to symmetry we can assume that $\mathcal{L}$ is parallel with the x-axis, i.e. $\mathcal{L} = \left\{\begin{pmatrix}1 & 0 & \dots & 0\end{pmatrix}^T\cdot\lambda + \begin{pmatrix}0 & a_2 & \dots & a_d\end{pmatrix}^T \mid \lambda\in\mathbb{R}\right\}$. By symmetry we can further assume $a_i \ge 0$ for all $2 \le i \le d$. We put for convenience $l := \sqrt{\sum_{i=2}^d a_i^2}$. We remark $\mathcal{C}_d(r_1, r_2)\cap\mathcal{L} = \emptyset$ for $l^2 > r_2^2$. The x-coordinates of the intersections of $\mathcal{L}$ with the d-dimensional sphere of radius r_1 are given by $\pm\sqrt{r_1^2 - l^2}$, as long as $l^2 \le r_1^2$. Similarly the x-coordinates of the intersections of $\mathcal{L}$ and the d-dimensional sphere of radius r_2 are given by $\pm\sqrt{r_2^2 - l^2}$, as long as $l^2 \le r_2^2$. For $l^2 \le r_1^2$ we have

$$\lambda_1\left(\mathcal{C}_d(r_1,r_2)\cap\mathcal{L}\right) = 2\cdot\underbrace{\left(\sqrt{r_2^2-\sum_{i=2}^d a_i^2}-\sqrt{r_1^2-\sum_{i=2}^d a_i^2}\right)}_{=:h_1(a_2,\dots,a_d)}.$$

Since

$$\frac{\partial h_1}{\partial a_i}(a_2,\ldots,a_d) = a_i \cdot \left(\frac{1}{\sqrt{r_1^2 - \sum\limits_{i=2}^{d} a_i^2}} - \frac{1}{\sqrt{r_2^2 - \sum\limits_{i=2}^{d} a_i^2}} \right) \geq 0$$

we can assume $l^2 \geq r_1^2$ for the maximum length of the intersection. If the a_i are restricted by an inequality $l^2 \leq k^2 \leq r_1^2$ the maximum length of the intersection is bounded from above by $2\sqrt{r_2^2 - k^2} - 2\sqrt{r_1^2 - k^2}$.

For $r_1^2 \leq \sum_{i=2}^{d} a_i^2 \leq r_2^2$ we have

$$\lambda_1\left(\mathcal{C}_d(r_1,r_2)\cap\mathcal{L}\right) = 2\cdot \underbrace{\sqrt{r_2^2 - \sum_{i=2}^{d} a_i^2}}_{=:h_2(a_2,\ldots,a_d)}$$

and

$$\frac{\partial h_1}{\partial a_2}(a_2,\ldots,a_d) = -a_i \cdot \frac{1}{\sqrt{r_2^2 - \sum\limits_{i=2}^{d} a_i^2}} \leq 0,$$

so that the extremal values is taken at $\sum_{i=2}^{d} a_i^2 = r_1^2$ where we have $\lambda_1\left(\mathcal{C}_d(r_1,r_2)\cap\mathcal{L}\right) \leq 2\sqrt{r_2^2 - r_1^2}$.

Thus for an arbitrary line $\mathcal{L}$ we have

$$\lambda_1\left(\cup_{n\geq 30}\mathcal{B}_n^d \cap \mathcal{L}\right) \leq \sum_{n=30}^{\infty} 2\sqrt{\left(1+\frac{1}{n^4}\right)^2 - 1^2} \leq \sum_{n=30}^{\infty} \frac{2\sqrt{3}}{n^2} < 0.12.$$

For the remaining part we restrict ourselves on lines being parallel to the x-axis. If $l < 30$ then

$$\begin{aligned}\lambda_1\left(\cup_{n\geq 30}\mathcal{A}_n^d \cap \mathcal{L}\right) &\leq 2\sqrt{\left(30+\frac{1}{d\cdot 30^d}\right)^2 - 30^2} + \sum_{n=31}^{\infty} 2\sqrt{\left(n+\frac{1}{dn^d}\right)^2 - l^2} - 2\sqrt{n^2-l^2} \\ &\leq 0.366 + 2\sum_{n=31}^{\infty} \frac{\frac{2}{n}}{2\sqrt{n^2-30^2}} < 0.47.\end{aligned}$$

For $l \geq 30$ we have

$$\begin{aligned}\lambda_1\left(\cup_{n\geq 30}\mathcal{A}_n^d \cap \mathcal{L}\right) &\leq 4\sqrt{\left(\lfloor l\rfloor+\frac{1}{d\cdot \lfloor l\rfloor^d}\right)^2 - \lfloor l\rfloor^2} + \sum_{n=\lfloor l+2\rfloor}^{\infty} 2\sqrt{\left(n+\frac{1}{dn^d}\right)^2 - l^2} - 2\sqrt{n^2-l^2} \\ &\leq 0.732 + +2\int_{\lfloor l+1\rfloor}^{\infty} \frac{1}{x\sqrt{x^2-l^2}}\,\mathrm{d}\,x \\ &= 0.732 + \frac{2}{l}\cdot \arcsin\left(\frac{l}{\lfloor l+1\rfloor}\right) \\ &\leq 0.732 + \frac{2}{l}\cdot\frac{\pi}{2} < 0.84.\end{aligned}$$

Since $0.12 + \max(0.47, 0.84) < 1$ we have $\lambda_1(\mathcal{P}\cap\mathcal{L}) < 1$ for each line $\mathcal{L}$. □

Since spherical symmetric slices S_d and caps C_d are very prominent objects in our constructions we provide some further expressions for their volumes at this place. To this end let us define $v(d) :=$

$\int_0^{\frac{\pi}{6}} \cos^d(x)\,\mathrm{d}\,x$. The first few values are given by:

$$
\begin{aligned}
v(1) &= \frac{1}{2}\\
v(2) &= \frac{1}{8}\cdot\sqrt{3}+\frac{1}{12}\cdot\pi\\
v(3) &= \frac{11}{24}\\
v(4) &= \frac{9}{64}\cdot\sqrt{3}+\frac{1}{16}\cdot\pi\\
v(5) &= \frac{203}{480}\\
v(6) &= \frac{9}{64}\cdot\sqrt{3}+\frac{5}{96}\cdot\pi\\
v(7) &= \frac{1759}{4480}\\
v(8) &= \frac{279}{2048}\cdot\sqrt{3}+\frac{35}{768}\cdot\pi\\
v(9) &= \frac{59123}{161280}\\
v(10) &= \frac{2673}{20480}\cdot\sqrt{3}+\frac{21}{512}\cdot\pi
\end{aligned}
$$

Using integration by parts we can immediately compute

$$
v(d)=\begin{cases} \frac{(2m-1)!!}{(2m)!!}\cdot\left(\frac{1}{2}\cdot\sum_{k=0}^{m-1}\frac{(2k)!!}{(2k+1)!!}\cdot\frac{\sqrt{3}}{2}\cdot\left(\frac{3}{4}\right)^k+\frac{\pi}{6}\right) & \text{for } d=2m,\\ \frac{(2m)!!}{(2m+1)!!}\cdot\frac{1}{2}\cdot\sum_{k=0}^{m}\frac{(2k-1)!!}{(2k)!!}\cdot\left(\frac{3}{4}\right)^k & \text{for } d=2m+1. \end{cases}
$$

Given the integer sequence A091814 from the "On-line encyclopedia of integer sequences" $v(d)$ can be written as $\frac{A091814(d))\cdot\left(\frac{d-1}{2}\right)!}{d!\cdot 2^{\frac{d+1}{2}}}$ for all odd d. Benoit Cloitre contributed the following second order recursion formula: $v(1)=\frac{1}{2}$, $v(3)=\frac{11}{24}$, and

$$
v(2n-1)=\frac{1}{8n-4}\cdot\Big((14n-17)\cdot v(2n-3)-6(n-2)\cdot v(2n-5)\Big)
$$

for $n\geq 3$. A similar recursion formula can be obtained for even d, where $v(d)$ can be written as $q(d)\cdot\sqrt{3}+\frac{\binom{d-1}{\frac{d}{2}}}{2^d\cdot 3}\cdot\pi$ for rational numbers $q(d)$.

Additionally one can compute the corresponding ordinary generating function:

$$
F(z):=\sum_{k=0}^{\infty}v(k)z^k=\sum_{k=0}^{\infty}\int_0^{\frac{\pi}{6}}(z\cos t)^k\,\mathrm{d}\,t=\int_0^{\frac{\pi}{6}}\frac{\mathrm{d}\,t}{1-z\cos t}=\frac{2}{\sqrt{1-z^2}}\arctan\left(\sqrt{\frac{1+z}{1-z}}\cdot\tan\frac{\pi}{12}\right).
$$

We will apply singularity analysis in order to determine the asymptotic behavior of $a_n:=F_\alpha(z)[z^n]$, where slightly more generally $F_\alpha(z):=\frac{2}{\sqrt{1-z^2}}\arctan\left(\sqrt{\frac{1+z}{1-z}}\cdot\alpha\right)$, see e.g. [20, chapter VI]. The main singularity is at $z=1$ since there is a compensation for $z=-1$. From

$$
\begin{aligned}
\arctan\left(\sqrt{\frac{1+z}{1-z}}\cdot\alpha\right) &= \frac{\pi}{2}+O\big((1-z)^{\frac{1}{2}}\big),\\
\frac{2}{\sqrt{1+z}} &= \sqrt{2}+O(1-z),\ \text{and}\\
[z^n]\frac{1}{\sqrt{1-z}} &= \frac{1}{\sqrt{\pi n}}+O\left(\frac{1}{n^{\frac{3}{2}}}\right)
\end{aligned}
$$

we conclude

$$a_n = \sqrt{\frac{\pi}{2n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

I.e. we have $v(d) \sim \sqrt{\frac{\pi}{2d}}$.

As mentioned in the introduction Ulam asked for a dense set in the plane so that all pairwise distances between the points are rational. We will demonstrate that there are dense sets so that all pairwise distances between points are irrational. The construction is based on an idea of Qiaochu Yuan, see http://math.stackexchange.com/questions/86805/is-there-a-dense-subset-of-mathbbr2-with-all-distances-being-incommensurab. Fix a dimension $d \geq 1$ and let R be a cocountable subset of $\mathbb{R}$, i.e. $\mathbb{R}\backslash R$ is countable. We will construct a countable dense subset S of $\mathbb{R}^d$ such that the ratio of any two distances between points in S, unequal to $\{s_1, s_2\}$ belongs to R. To do this, begin by placing two points a unit distance apart. Now enumerate the balls with rational center and rational radius in $\mathbb{R}^d$ and place points s_n in the interior of each such ball in turn satisfying the given condition. This is always possible because the set of all points at which s_n cannot be placed is a countable union of sets of measure zero (one for each possible ratio of two distances lying outside of), hence has measure zero. Choosing R as the set of irrational numbers, which is cocountable, and deleting s_1 from S gives a dense subset of $\mathbb{R}^d$ without pairs of points having a rational distance. Similarly one can construct countable dense sets avoiding points with distance in any countable subset of $\mathbb{R}$, for instance in $\mathbb{Q}(\sqrt{2})$.

The asymptotic volumes of the p-gon construction can be computed using integrals. For dimension $d = 2$ the components are given by polygons $\mathcal{K}$, whose (signed) area can be computed either by evaluating $\int_{\mathcal{K}} 1 \,\mathrm{d}\,u$ or by $\frac{1}{2}\sum\limits_{i=0}^{n-1} x_iy_{i+1} - x_{i+1}y_i$, where $(x_1, y_1), \ldots, (x_n, y_n) = (x_0, y_0)$ are the coordinates of the vertices of $\mathcal{K}$ in counterclockwise order, see e.g. [27, 40]. The first formula gets useful if one considers the construction in dimension $d = 3$. Here the volume is given by $2 \cdot \int_{\mathcal{K}} \sqrt{\frac{1}{4} - x^2 - y^2}\,\mathrm{d}(x, y)$. One way to evaluate an integral over a polygonal domain $\mathcal{K}$ is to dissect the domain into triangles, i.e.

$$2 \cdot \int\limits_{\mathcal{K}} \sqrt{\frac{1}{4} - x^2 - y^2}\,\mathrm{d}(x, y) = 2 \cdot \sum_{i=0}^{n-1} \int\limits_{\Delta\left((x_i,y_i),(x_{i+1},y_{i+1}),(0,0)\right)} \sqrt{\frac{1}{4} - x^2 - y^2}\,\mathrm{d}(x, y),$$

where $\Delta(p_1, p_2, p_3)$ denotes the triangular domain spanned by the points p_1, p_2, p_3. So let us now suppose that the triangular domain $\mathcal{T}$ is spanned by the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3). We consider the affine transformations $x(u, v) = x_1 + u(x_2 - x_1) + v(x_3 - x_1)$, $y(u, v) = y_1 + u(y_2 - y_1) + v(y_3 - y_1)$, whose determinant of the corresponding Jacobian matrix is given by

$$|J| := \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

The affine transformations maps $\mathcal{T}$ to the triangle spanned by $(0, 0)$, $(1, 0)$, and $(0, 1)$, i.e. it maps $(0, 0)$ to (x_1, y_1), $(1, 0)$ to (x_2, y_2), and $(0, 1)$ to (x_3, y_3), so that we have

$$2 \cdot \int\limits_{\mathcal{T}} \sqrt{\frac{1}{4} - x^2 - y^2}\,\mathrm{d}(x, y) = 2|J| \cdot \int\limits_0^1 \int\limits_0^{1-u} \sqrt{\frac{1}{4} - x(u, v)^2 - y(u, v)^2}\,\mathrm{d}\,v\,\mathrm{d}\,u.$$

Once the corners of the polygonal domain are determined the remaining computation can be performed automatically by a computer algebra package like e.g. `Maple`.

Let $p \geq 5$ be a prime. After scaling we may assume that the centers of the p circles are located at $(x_i, y_i) := \big(\sin(i\alpha) - 1, \cos(i\alpha)\big)$, where $0 \leq i \leq p - 1$ and $\alpha = \frac{2\pi}{p}$, i.e. $(x_0, y_0) = (0, 0)$. The two cutting lines arising from circle i, with $1 \leq i \leq p - 1$, are given by

$$\left\{ \begin{pmatrix} -\cos(i\alpha) \\ \sin(i\alpha) - 1 \end{pmatrix} \cdot t \pm \frac{1}{4(2 - 2\sin(i\alpha))} \begin{pmatrix} \sin(i\alpha) - 1 \\ \cos(i\alpha) \end{pmatrix} \,\Big|\, t \in \mathbb{R} \right\}.$$

The numerical coordinates of the vertices of the octagon for $n = p = 5$, $d = 2$ are given by

$$\begin{aligned}
(x_1, y_1) &= (-0.21266270208801, 0.154508497187474) \\
(x_2, y_2) &= (-0.262865556059567, -1.03568412526678e - 16) \\
(x_3, y_3) &= (-0.21266270208801, -0.154508497187474) \\
(x_4, y_4) &= (0, -0.309016994374947) \\
(x_5, y_5) &= (0.21266270208801, -0.154508497187474) \\
(x_6, y_6) &= (0.262865556059567, -3.39829618438425e - 16) \\
(x_7, y_7) &= (0.21266270208801, 0.154508497187473) \\
(x_8, y_8) &= (0, 0.309016994374948)
\end{aligned}$$

Inserting this in one of the two stated formulas yields an area of approximately 1.0633135 for dimension $d = 2$ and a volume of approximately 0.9867390 for dimension $d = 3$.

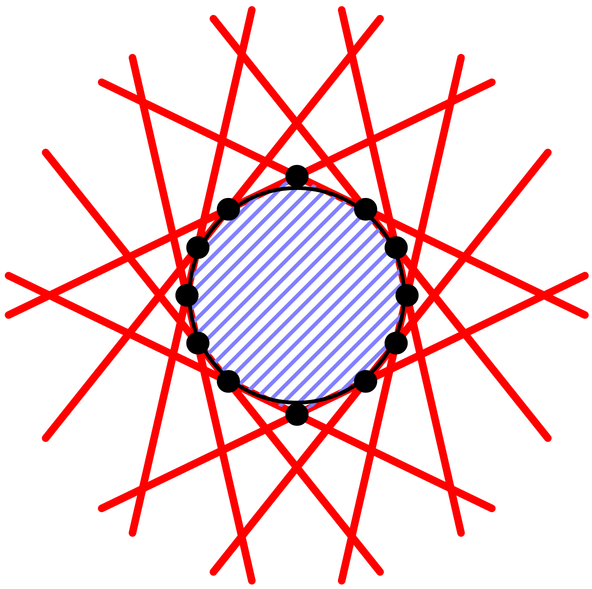

Figure 12: 7-gon construction: One of the resulting dodecagons.

We would like to remark that the p-gon construction for $d = 2$ and $p = 3$ yields an asymptotic area of $\sqrt{3}2 \approx 0.8660$, see Figure 14 and Figure 15. Although the exact p-gon construction is somewhat technically complicated to evaluate it can provide a more accessible lower bound. Cutting from all continuous directions yields a circle of diameter $\frac{1}{2}$ in dimension two so that the asymptotic area is larger then $n \cdot \frac{\pi}{4}$. In higher dimensions we can intersect the d-dimensional unit balls with cylinders of diameter $\frac{1}{2}$.

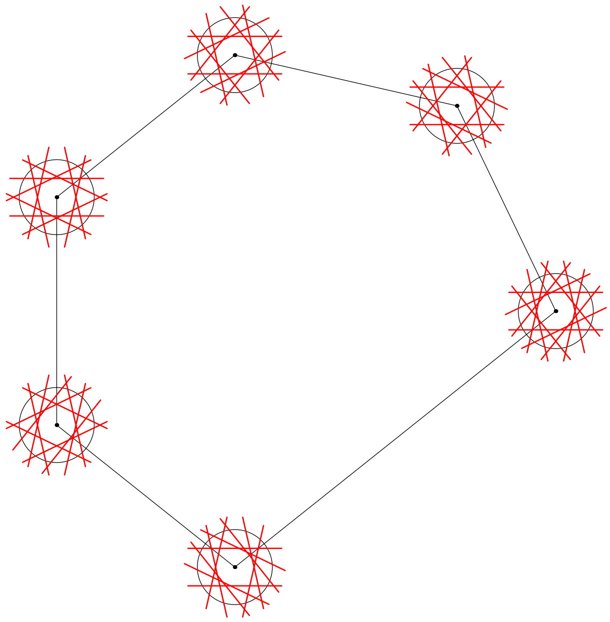

Figure 13: 7-gon construction for $d = 2$ and $n = 6$.

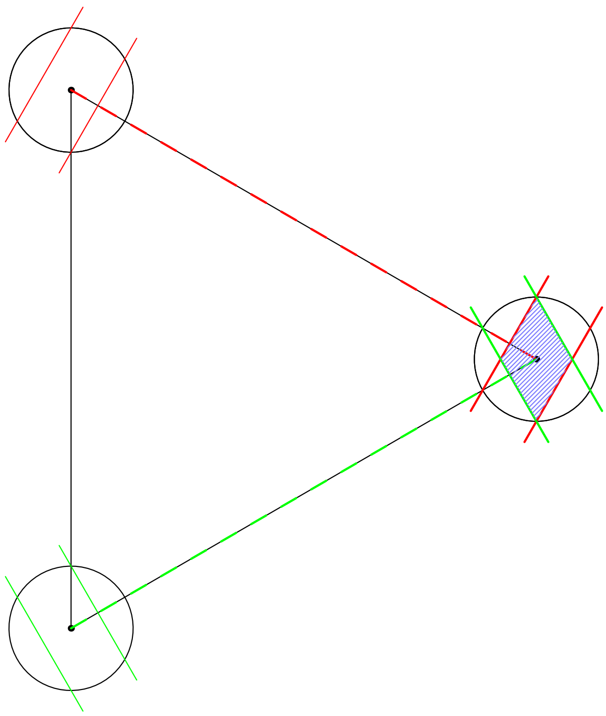

Figure 14: p-gon construction: Integral distance avoiding point set for $d = 2$ and $p = n = 3$.

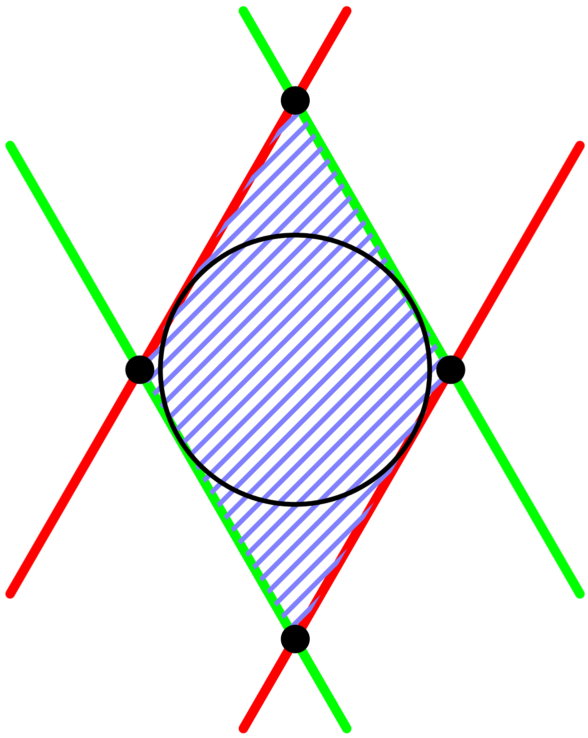

Figure 15: 3-gon construction: One of the resulting parallelograms.